UMN-TH-1331/95
hep-th/9505113
June 1, 1995 (corrected)
To appear in Phys. Lett. B

Spontaneous Magnetization in Lorentz Invariant Theories

Denne Wesolowski and Yutaka Hosotani

School of Physics and Astronomy, University of Minnesota
Minneapolis, Minnesota 55455, U.S.A.

wesolowski@mnhep.hep.umn.edu
yutaka@mnhep.hep.umn.edu

Abstract

In a class of three-dimensional Abelian gauge theories with both light and heavy fermions, heavy chiral fermions can trigger dynamical generation of a magnetic field, leading to the spontaneous breaking of the Lorentz invaiance. Finite masses of light fermions tend to restore the Lorentz invariance.

It has been shown in refs. [1, 2] that in a variety of classes of three-dimensional gauge theories described by

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\kappa_0}{2}\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho \\ &\quad + \sum_a \frac{1}{2}\left[\,\overline{\psi}_a\,,\,(\gamma_a^\mu(i\partial_\mu + q_a A_\mu) - m_a)\psi_a\,\right]\,, \end{aligned} \tag{1}$$

the Lorentz invariance of the vacuum is spontaneously broken via dynamical generation of a magnetic field. In other words, it has been established that for these models there exists a variational ground state, in which $\langle\, F_{12}(x)\,\rangle = -B \neq 0$, that has a lower energy density than the perturbative ground state. Quantum fluctuations play a crucial role in lowering the energy density for these states.[1] In the previous papers models with $\kappa_0 \neq 0$ and $m_a \ll q_a^2$ were considered.

There are two crucial ingredients. First, once a magnetic field is dynamically generated, the fermion energy spectrum is characterized by that of Landau levels. The lowest level has an energy $\epsilon_{a0} = m_a$. In the massless fermion limit $\epsilon_{a0} = 0$ so that a perturbative ground state is infinitely degenerate. This degeneracy is lifted by interactions. With certain conditions satisfied, a variational state, in which the lowest Landau levels of chirality $+$ fermions are completely occupied, whereas those of chirality $-$ fermions are empty, has the lowest energy.[1] Secondly it is recognized that the lowering of the energy density by a non-vanishing magnetic field is mainly due to the shift in zero-point energies of photons. If $\kappa_0 \neq 0$, a photon is topologically massive in perturbation theory. Once $B \neq 0$ is generated, a photon becomes a gapless mode, serving as the Nambu-Goldstone boson associated with the spontaneous breaking of the Lorentz invariance. Hence zero-point energies are shifted, resulting in lowering the energy density of the ground state.[2, 4]

Recently Gusynin, Miransky, and Shovkovy have shown that in a model with an external magnetic field and an additional four-fermi interaction, flavor-nonsinglet condensate $\langle\, \overline{\psi}_a\psi_a\,\rangle$ can be spontaneously generated.[5, 6] Again the existence of infinitely degenerate perturbative ground states in the massless fermion limit plays a crucial role.

We here stay within renormalizable theories described by (1) with general fermion

[1] Cea has considered a variational ground state with $B \neq 0$, which, however, differs from ours.[3]

content to obtain a criterion for the spontaneous magnetization ($B \neq 0$) to occur. We shall also evaluate contributions to the energy density due to finite fermion masses.

A brief review is in order. In (1) ψ_a is a two-component Dirac spinor. These spinors, in $2+1$ dimensions, are characterized by the signature of the two-dimensional Dirac matrices via the chirality $\eta_a = \frac{i}{2}\,\mathrm{Tr}\,\gamma_a^0\gamma_a^1\gamma_a^2 = \pm 1$. We utilize the representation $\gamma_a^\mu = (\eta_a\sigma_3, i\sigma_1, i\sigma_2)$. Since the model is invariant under charge conjugation and the transformation $m_a \to -m_a$ is equivalent to the transformation $\gamma_a^\mu \to -\gamma_a^\mu$ or $\eta_a \to -\eta_a$, one can take $q_a > 0$ and $m_a \geq 0$ without loss of generality.

As in [1] we have the expression for the energy density

$$
\begin{aligned}
\Delta\mathcal{E} &= \mathcal{E}(B \neq 0) - \mathcal{E}(B = 0) \\
&= \frac{1}{2}B^2 + \Delta\mathcal{E}_{\text{f.z.e.}} + \Delta\mathcal{E}_{\text{fluct}}
\end{aligned}
$$

$$
\begin{aligned}
\Delta\mathcal{E}_{\text{fluct}} &\sim \Delta\mathcal{E}_{\text{RPA}} \\
&= -\frac{i}{2}\int \frac{d^3p}{(2\pi)^3}\,\Big\{\, \mathrm{tr}\,\ln\left(1 - \Gamma^{(2)}D_0\right)\big|_{qB,\nu} - (B \to 0)\,\Big\} \\
&= -\frac{i}{2}\int \frac{d^3p}{(2\pi)^3}\,\ln \frac{(1+\Pi_0)\Big\{1 + \frac{1}{p^2}(p_0^2\Pi_0 - \vec{p}^{\,2}\Pi_2)\Big\} - \frac{1}{p^2}(\kappa_0 - \Pi_1)^2}{(B \to 0)} \quad . \qquad (2)
\end{aligned}
$$

$\Delta\mathcal{E}_{\text{f.z.e.}}$ is the shift in fermion zero-point energies in a given uniform magnetic field. $\Delta\mathcal{E}_{\text{fluct}}$ represents the shift in the energy density due to quantum fluctuations of fields, and is approximated by the RPA correction $\Delta\mathcal{E}_{\text{RPA}}$. The vertex functions Γ, the bare and full photon propagators D_0 and D, and the invariant functions Π's are related by

$$
\begin{aligned}
\Gamma^{\mu\nu}(p) &= (D_0^{-1} - D^{-1})^{\mu\nu} \\
&= (p^\mu p^\nu - p^2 g^{\mu\nu})\Pi_0 + i\epsilon^{\mu\nu\rho}p_\rho\Pi_1 \\
&\quad + (1-\delta^{\mu 0})(1-\delta^{\nu 0})(p^\mu p^\nu - \vec{p}^{\,2}\delta^{\mu\nu})(\Pi_2 - \Pi_0). \qquad (3)
\end{aligned}
$$

The superscript for Γ in (2) indicates that $\Gamma(p)$ is approximated by $\mathrm{O}(e^2)$ diagrams.

Much as in [1, 2] we obtain a consistency condition which the fermion content must satisfy in order that there be symmetry breaking. Equations of motion imply, in the true vacuum, that $\kappa_0\langle F_{12}\rangle = \langle J^0\rangle$. Since $\langle J^0\rangle = \Pi_1(0)\langle F_{12}\rangle$, the relation $\kappa_0 = \Pi_1(0)$ must be satisfied in order for $B = -\langle F_{12}\rangle$ to be nonvanishing. This consistency condition

was shown to relate to the Nambu-Goldstone theorem associated with the spontaneous breaking of the Lorentz invariance. The dynamically generated nonzero magnetic field must accompany a gapless excitation, which is the photon of the theory.[2] Since the induced Chern-Simons term has a coefficient $-\Pi_1(0)$, one can phrase that the bare Chern-Simons term must be exactly cancelled by the induced Chern-Simons term in order to have B$\neq$ 0.

We now consider some definite cases of fermion content. Suppose that there are light fermions ($m_{\rm light} \ll q^2$) and heavy fermions ($m_{\rm heavy} \gg q^2$). In perturbation theory

$$\Pi_0|_{B=0} = \Pi_2|_{B=0} = \sum \frac{q_a^2}{8\pi}\frac{1}{(-p^2)^{1/2}}\left(\sqrt{z_a} + (1-z_a)\sin^{-1}\frac{1}{\sqrt{1+z_a}}\right)$$
$$\Pi_1(p)|_{B=0} = -\sum \eta_a \frac{q_a^2}{4\pi}\sqrt{z_a}\,\sin^{-1}\frac{1}{\sqrt{1+z_a}}\ , \quad z_a = \frac{4m_a^2}{-p^2}\ . \tag{4}$$

This gives, for the part of heavy fermion contributions,[11, 12, 13]

$$\Pi_1^{\rm heavy}(0)|_{B=0} = -\sum_{\rm heavy}\eta_a\frac{q_a^2}{4\pi} = \kappa_{\rm heavy}\ . \tag{5}$$

In the presence of $B \neq 0$ the energy spectrum of fermions has the structure of Landau levels $\epsilon_{an} = \sqrt{m_a^2 + 2nq_a|B|}$ ($n = 0, 1, 2, \cdots$). There is asymmetry in the $n = 0$ modes. $\Pi_1(0)$ is found to be

$$\Pi_1(0)\Big|_{B\neq0} = \frac{1}{2\pi}\sum \eta_a q_a^2(\nu_a - \frac{1}{2})\ . \tag{6}$$

Here ν_a is the filling fraction of the Lowest Landau level. We suppose that either $\nu_a = 0$ (empty) or $\nu_a = 1$ (completely filled).

The lowest Landau level has an energy m_a. Hence for heavy fermions we may assume that the lowest Landau levels are completely empty, and assign a filling factor $\nu_a = 0$. The approximation is valid for $|B| \ll m_{\rm heavy}^{3/2}$. Comparing (6) with (5), one finds

$$\Pi_1^{\rm heavy}(0)\Big|_{B\neq0} = \Pi_1^{\rm heavy}(0)|_{B=0} \equiv -\kappa_{\rm heavy} \tag{7}$$

Further heavy fermion contributions to Π_0 and Π_2 at low energies ($|p^2| \ll m_{\rm heavy}^2$) are $O(q^2/m_{\rm heavy})$. In other words, the only effect of heavy fermions at low energies is to shift the bare Chern-Simons coefficient κ_0 to $\kappa_0 + \kappa_{\rm heavy}$.

[2] The possibility of regarding a photon as a Nambu-Goldstone boson has been discussed by Bjorken[8], by Nambu[9], and by Kovner and Rosenstein[10]. Our picture is different from theirs. In our case the Lorentz invariance is spontaneously broken in the physical content.

The variational ground state (true vacuum) is specified with $\{\nu_a \ ; \ a \in \text{light fermions}\}$. The consistency condition to be satisfied is thus

$$\kappa_0 + \kappa_{\rm heavy} = \frac{1}{2\pi} \sum_{\rm light} \eta_a q_a^2 (\nu_a - \frac{1}{2}) \quad . \tag{8}$$

Even if the bare Chern-Simons coeffiecient $\kappa_0 = 0$, the requisite consistency condition can be satisfied.

To be more definite, we consider the following models:

[Model A] The light fermion content is chirally symmetric. In other words, light fermions come in a pair of $(m_a, q_a, \eta_a = \pm)$. $\{\nu_a\}$ must satisfy (8).

[Model B] This is a special case of model A, in which $\nu_a = 1$ or 0 for $\eta_a = +$ or $-$, respectively. In this case the consistency condition is reduced to

$$\kappa_0 + \kappa_{\rm heavy} = \frac{1}{4\pi} \sum_{\rm light} q_a^2 \quad . \tag{9}$$

[Model C] In model B we suppose that $q_a = q$ and $m_a = m$ $(a = 1 \sim N_f)$. The consistency condition is

$$\kappa_0 + \kappa_{\rm heavy} = \frac{N_f q^2}{4\pi} \quad . \tag{10}$$

The shift in zero-point energies of photons between the perturbative vacuum and the variational ground state is given by [2]

$$\begin{aligned} \Delta\mathcal{E} &= \int \frac{d\vec{p}}{(2\pi)^2} \, \frac{1}{2} \Big\{ \omega(\vec{p})_{B\neq 0} - \omega(\vec{p})_{B=0} \Big\} \\ &\sim \int \frac{d\vec{p}}{(2\pi)^2} \, \frac{1}{2} \Big\{ \sqrt{\vec{p}^{\,2} + \kappa_{tot}^2(p)_{B\neq 0}} - \sqrt{\vec{p}^{\,2} + \kappa_{\rm tot}^2(p)_{B=0}} \Big\} \ , \end{aligned} \tag{11}$$

where $\kappa_{\rm tot}(p) = \kappa_0 - \Pi_1(p)$. In model A $\Pi_1(p)^{\rm light} = 0$ identically to $\mathrm{O}(e^2)$. Hence, for $p^2 \ll m_{\rm heavy}^2$, $\kappa_{\rm tot}(p)_{B=0} \sim \kappa_0 + \kappa_{\rm heavy}$. On the other hand the consistency condition implies that $\kappa_{\rm tot}(0)_{B\neq 0} = 0$. For large $|\vec{p}\,|$, $\kappa_{\rm tot}(p)_{B\neq 0} \sim \kappa_{\rm tot}(p)_{B=0}$. The deviation develops below $l_{\rm ave}^{-1}$ where $l_{\rm ave}$ is a characteristic magnetic length. For $m_{\rm light} = 0$ [2]

$$\begin{aligned} \frac{1}{l_{\rm ave}^2} &= \frac{|\sum_{\rm light} \eta_a \nu_a q_a^3|}{|\sum_{\rm light} \eta_a \nu_a q_a^2|} \times |B| \qquad \text{(model A)} \\ &= \frac{\sum_{\rm light} q_a^3}{\sum_{\rm light} q_a^2} \times |B| \qquad \text{(model B)} \ . \end{aligned} \tag{12}$$

To get an estimate of (11), we approximate $\kappa_{\rm tot}(p)_{B\neq 0}=0$ and $\kappa_{\rm tot}(p)_{B=0}=\kappa_0+\kappa_{\rm heavy}$ for $|\vec{p}\,|\le l_{\rm ave}^{-1}$, whereas $\kappa_{\rm tot}(p)_{B\neq 0}=\kappa_{\rm tot}(p)_{B=0}$ for $|\vec{p}\,|\ge l_{\rm ave}^{-1}$. Then

$$\begin{aligned}\Delta\mathcal{E} &\sim -\frac{1}{8\pi}\frac{|\kappa_0+\kappa_{\rm heavy}|}{l_{\rm ave}^2}+{\rm O}(l_{ave}^{-3})\\ &= -\frac{1}{32\pi^2}\sum_{\rm light} q_a^3\cdot|B| \qquad \text{(model B)}\end{aligned} \tag{13}$$

for $|\kappa_0+\kappa_{\rm heavy}|\,l_{\rm ave}\gg 1$. As (11) and (13) are negative definite, the energy density $\mathcal{E}$ is indeed lowered for the vacuum state with $B\neq 0$ provided that $\kappa_0+\kappa_{\rm heavy}$, $\sum_{\rm light}\eta_a\nu_a q_a^2$, and $\sum_{\rm light}\eta_a\nu_a q_a^3$ are all non-vanishing and that the consistency condition (8) is satisfied. In particular, even if the bare Chern-Simons term is absent ($\kappa_0=0$), heavy fermions can induce a dynamical magnetic field $B\neq 0$.

The same conclusion is obtained more convincingly by evaluating (2). We recall [1]

$$\begin{aligned}&\Pi_0(p)|_{B\neq 0}-\Pi_0(p)|_{B=0}\\ &\qquad=\sum_{\rm light}\nu_a\frac{q_a^2}{2\pi}\frac{1}{p_0}\Big\{\frac{1}{2-p^2l_a^2-2m_ap_0l_a^2}-\frac{1}{2-p^2l_a^2+2m_ap_0l_a^2}\Big\}+{\rm O}(B^2)\\ &\Pi_1(p)|_{B\neq 0}-\Pi_1(p)|_{B=0}\\ &\qquad=\sum_{\rm light}\eta_a\nu_a\,\frac{q_a^2}{2\pi}\Big\{\frac{1}{2-p^2l_a^2-2m_ap_0l_a^2}+\frac{1}{2-p^2l_a^2+2m_ap_0l_a^2}\Big\}+{\rm O}(B^2)\\ &\Pi_2(p)|_{B\neq 0}-\Pi_2(p)|_{B=0}={\rm O}(B^2)\end{aligned} \tag{14}$$

where $l_a^{-2}=|q_aB|$. When $m_a=0$, $\Pi_0(p)_{B=0}=\Pi_2(p)_{B=0}=(\sum_{\rm light}q_a^2/16)(-p^2)^{-1/2}$ and $\Pi_1(p)_{B=0}=-\kappa_{\rm heavy}$ for $|-p^2|\ll m_{\rm heavy}^2$. Inserting these with (14) into (11) gives the result [2]

$$\begin{aligned}\Delta\mathcal{E}_{\rm RPA} &= -\frac{\sum_{\rm light}\eta_a\nu_a q_a^3}{2\pi^3}\cdot\tan^{-1}\frac{8\sum_{\rm light}\eta_a\nu_a q_a^2}{\pi\sum_{\rm light}q_a^2}\cdot|B|+{\rm O}(|B|^{3/2})\\ &= -\frac{1}{4\pi^3}\tan^{-1}\frac{4}{\pi}\sum_{\rm light}q_a^3\cdot|B|+{\rm O}(|B|^{3/2}) \qquad \text{(model B).}\end{aligned} \tag{15}$$

Comparing this with (13), one finds that the shift in zero-point energies accounts for half of the RPA effect.

Recently Cangemi, D'Hoker and Dunne have evaluated the effective potential in a magnetic field, supposing that all Landau levels are empty in the vacuum.[7] They found

that a uniform constant magnetic field configuration shows instability against forming inhomogeneity. Cangemi et al.'s variational vacuum state corresponds to $\{\nu_a = 0\}$ whereas in our variational vacuum $\nu_a = 1$ for $\eta_a = +$. It is desirable to evaluate the effective potential for a general vacuum state specified with $\{\nu_a\}$.

In passing, if $\nu_a = 0$ in model A, the consistency condition (8) becomes $\kappa_0 + \kappa_{\rm heavy} = -(4\pi)^{-1} \sum \eta_a q_a^2 = 0$. Further, from the symmetry $\Pi_1(p)_{\rm light} = 0$. Hence both $\kappa_{\rm tot}(p)_{B=0}$ and $\kappa_{\rm tot}(p)_{B\neq 0}$ vanish for $p^2 \ll m_{\rm heavy}^2$. In other words the state with $(B \neq 0, \nu_a = 0)$ has a higher energy density than the state with $B = 0$.

It is straightforward to evaluate $\mathrm{O}(m_a)$ corrections to $\Delta\mathcal{E}$. We evaluate them in model C, supposing $q_a = q, m_a = m \neq 0$. $\Delta\mathcal{E}/q^6$ is a function of two dimensionless parameters m/q^2 and $|B|/q^3$. We shall find an expression for $\Delta\mathcal{E}$ valid for $m^2/q^4 \ll |B|/q^3 \ll 1$. (Note $(ml)^2 = m^2/(q|B|) \ll 1$.)

Corrections to $\Delta\mathcal{E}_{\rm RPA}$ are found from (2) and (14). Π_0 and Π_1 give corrections of $\mathrm{O}(m)$ and $\mathrm{O}(m^2)$, respectively. Hence a dominant correction comes from Π_0 and is found to be

$$\begin{aligned}
\Delta\mathcal{E}_{\rm RPA}^{\rm mass} &= \frac{N_f^2 q^4 m}{48\pi^3}\, f(y) \qquad \text{where} \ \ y = \Big(\frac{16}{N_f}\Big)^2 \frac{|B|}{q^3} \qquad \text{(model C)} \\
f(y) &= y \int_0^\infty dk\, \frac{k^3(\sqrt{y}\,k+1)}{(\sqrt{y}\,k+1)^2(k^2+2)^2 + (16k^4/\pi^2)} \\
&\sim \frac{1}{2(1+16/\pi^2)}\, y \left\{ -\ln y + 1.8 \right\} \qquad \text{for} \ \ y < 0.1 \ \ .
\end{aligned} \tag{16}$$

Notice that $\Delta\mathcal{E}_{\rm RPA}^{\rm mass} > 0$.

Secondly the lowest Landau level has an energy m_a so that a contibution from occupied levels is

$$\Delta\mathcal{E}_{\rm occupied}^{\rm mass} = \sum_{\rm light} \frac{\nu_a m_a}{2\pi l_a^2} = \frac{1}{2\pi} \sum_{\rm light} \nu_a m_a q_a\, |B| \ . \tag{17}$$

It is positive.

There is another important contribution coming from a shift in fermion zero-point energies. Landau levels are given by $\epsilon_{an} = \sqrt{m_a^2 + (2n/l_a^2)}$. Recalling that there is asymmetry in the lowest level, i.e. the $n = 0$ level exists for either a particle or an anti-particle,

one finds that fermion zero-point energies in the presence of a uniform magnetic field are

$$\Delta\mathcal{E}_{\text{f.z.e.}} = \sum_{\text{light}} \frac{1}{2\pi l_a^2}\Big\{ -\frac{1}{2}m_a - \sum_{n=1}^{\infty}\sqrt{m_a^2+\frac{2n}{l_a^2}}\Big\} \ . \tag{18}$$

To evaluate this sum we utilize the standard zeta-function technique, taking

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{\lambda_n^s} = \sum_n \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}e^{-\lambda_n t}. \tag{19}$$

Here we have $\lambda_n \equiv m^2+(2n/l^2)$ and need to calculate $\zeta(-\frac{1}{2})$. The substitution $u\equiv 2nt/l^2$ immediately leads to

$$\begin{aligned}\zeta(s) &= \Big(\frac{l^2}{2}\Big)^s \frac{1}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{1}{n^s}\int_0^\infty du\, u^{s-1}\exp\Big(-u-\frac{m^2l^2}{2n}u\Big)\\ &= \Big(\frac{l^2}{2}\Big)^s\zeta_R(s) - m^2\Big(\frac{l^2}{2}\Big)^{s+1}\zeta_R(s+1)+\cdots \ .\end{aligned} \tag{20}$$

Here ζ_R is the Riemann zeta function and the expansion is valid for $m^2l^2 \ll 1$. In particular

$$\zeta(-\tfrac{1}{2}) = -\frac{1}{4\pi}\zeta_R(\tfrac{3}{2})\Big(\frac{l^2}{2}\Big)^{-1/2} - m^2\Big(\frac{l^2}{2}\Big)^{1/2}\zeta_R(\tfrac{1}{2})+\cdots \ . \tag{21}$$

Note $\zeta_R(\frac{3}{2}) = 2.61238$ and $\zeta_R(\frac{1}{2}) = -1.46035 < 0$.

Utilizing (21) in (18), one finds

$$\begin{aligned}\Delta\mathcal{E}_{\text{f.z.e.}} &= \sum_{\text{light}}\frac{1}{4\sqrt{2}\pi^2}\zeta_R(\tfrac{3}{2})\frac{1}{l_a^3} - \sum_{\text{light}}\frac{m_a}{4\pi l_a^2} + \mathrm{O}(m^2)\\ &= \frac{1}{4\sqrt{2}\pi^2}\zeta_R(\tfrac{3}{2})\sum_{\text{light}} q_a^{3/2}|B|^{3/2} - \frac{1}{4\pi}\sum_{\text{light}} m_a q_a|B| + \mathrm{O}(m^2)\\ &\qquad \text{for } \frac{m^2}{q^4} \ll \frac{|B|}{q^3} \ll 1\end{aligned} \tag{22}$$

Notice that the second term (linear in m_a) is negative. This behavior has been noticed by Gusynin, Miransky, and Shovkovy in a different context.[5] (17) and (22) combine to give

$$\begin{aligned}&\Delta\mathcal{E}^{\text{mass}}_{\text{occupied}} + \Delta\mathcal{E}_{\text{f.z.e.}}\\ &= \frac{1}{4\sqrt{2}\pi^2}\zeta_R(\tfrac{3}{2})\sum_{\text{light}} q_a^{3/2}|B|^{3/2} + \frac{1}{2\pi}\sum_{\text{light}}(\nu_a - \tfrac{1}{2})m_a q_a|B| + \mathrm{O}(m^2) \ .\end{aligned} \tag{23}$$

We recognize that in model B the second term vanishes.

Combining the results of (15), (16), and (23), one finds, in model C,

$$\begin{aligned}
\frac{1}{q^6}\,\Delta\mathcal{E} &= \frac{1}{2}\frac{B^2}{q^6} - \frac{N_f}{4\pi^3}\tan^{-1}\frac{4}{\pi}\cdot\frac{|B|}{q^3} + \frac{N_f\zeta_R(\frac{3}{2})}{4\sqrt{2}\pi^2}\cdot\Big(\frac{|B|}{q^3}\Big)^{3/2} \\
&\qquad + \frac{N_f^2 m}{48\pi^3 q^2}\,f\Big(\frac{256|B|}{N_f^2 q^3}\Big) + \mathrm{O}(m^2, B^2) \\
&\sim \frac{1}{2}\Big(\frac{N_f}{16}\Big)^2 y^2 - 0.117\Big(\frac{N_f}{16}\Big)^3 y + 0.749\Big(\frac{N_f}{16}\Big)^4 y^{3/2} \\
&\qquad + 0.0328\Big(\frac{N_f}{16}\Big)^2\frac{m}{q^2}\,y(-\ln y + 1.8) + \mathrm{O}(m^2) \\
&\qquad\qquad \text{for } \frac{m^2}{q^4} \ll y = \Big(\frac{16}{N_f}\Big)^2\frac{|B|}{q^3} < 0.1
\end{aligned} \tag{24}$$

We have restored the Maxwell energy term $\frac{1}{2}B^2$. Other $\mathrm{O}(B^2)$ corrections have much smaller coefficients.

We observe that finite masses of light fermions tend to restore the Lorentz symmetry. $\Delta\mathcal{E}(y)$ develops a minimum at $y_{\rm min} \neq 0$. As m gets bigger, $y_{\rm min}$ decreases and $\Delta\mathcal{E}(y_{\rm min})$ increases. More specifically

	m/q^2	$y_{\rm min}$	$\Delta\mathcal{E}_{\rm min}/q^6$	$(ml)^2\vert_{\rm min}$
$N_f = 16$	0.	9.2×10^{-3}	-3.7×10^{-4}	
	0.02	8.7×10^{-3}	-3.3×10^{-4}	0.05
$N_f = 4$	0.	1.0×10^{-1}	-7.1×10^{-5}	
	0.02	7.2×10^{-2}	-3.5×10^{-5}	0.09

(25)

For larger values of m/q^2 the expressions (21) and (24) are not accurate.

In this paper we have shown that heavy fermions can induce spontaneous magnetization through the induced Chern-Simons coefficient at low energies. Further we showed that nonvanishing masses of light fermions tend to restore the Lorentz symmetry by giving a positive slope in the $|B|$ dependence near the minimum of $\Delta\mathcal{E}$.

Our consideration here was limited to the dependence of the energy density on a dynamically generated magnetic field. In view of the recent result by Gusynin et al. [5] it

is of great interest to find the effective action as a function of both dynamically generated magnetic field B and chiral condensate $\langle \overline{\psi}_a \psi_a \rangle$. It may be that both B and $\langle \overline{\psi}_a \psi_a \rangle$ are generated dynamically in a cooperative manner. We shall come back to this point in the near future.

Note added: Recently Kanemura and Matsushita have examined the behaviour of the model (1) at finite temperature in the massless fermion limit.[14] They have found that the coefficient of the linear term $\propto |B|$ in the energy density (15) or (24) remains negative even at finite temperature.

Acknowledgements

This work was supported in part by the U.S. Department of Energy under contract no. DE-AC02-83ER-40105. D.W. would like to thank C.L. Erickson and L.D. Reed for many stimulating and useful discussions.

References

[1] Y. Hosotani, Phys. Lett. 319B (1993) 332.

[2] Y. Hosotani, Phys. Rev. D51 (1995) 2022.

[3] P. Cea, Phys. Rev. D32 (1985) 2785.

[4] Y. Hosotani, UMN-TH-1304/94 (`hep-th/9407188`); UMN-TH-1308/94 (`hep-th/9408148`).

[5] V.P. Gusynin, V.A. Miransky, and I.A. Shovkovy, Phys. Rev. Lett. 73 (1994) 3499; NSF-ITP-94-74 (`hep-th/9407168`); Kiev-ITP-95-02E (`hep-ph/9501304`).

[6] R.R. Parwani, IP/BBSR/95-12 (`hep-th/9504020`).

[7] D. Cangemi, E. D'Hoker and G. Dunne, Phys. Rev. D51 (1995) 2513.

[8] J.D. Bjorken, Ann. Phys. A30 (1963) 174.

[9] Y. Nambu, Supplement of Prog. Theoret. Phys. Extra Number (1968) 190.

[10] A. Kovner and B. Rosenstein, Int. J. Mod. Phys. A30 (1992) 7419.

[11] A.N. Redlich, Phys. Rev. Lett. 52 (1984) 18; Phys. Rev. D29 (1984) 2366.

[12] K. Ishikawa, Phys. Rev. Lett. 53 (1984) 1615; Phys. Rev. D31 (1985) 1432.

[13] V.Y. Zeitlin, Yad. Fiz. 49 (1989) 742 [Sov. J. Nucl. Phys. 49 (1989) 461].

[14] S. Kanemura and T. Matsushita, OU-HET 212 (hep-th/9505146).